2801. In completed-square form, the parabolae may be written

$$\begin{aligned} y &= (x+a)^2 + b, \\ y &= (x+c)^2 + d. \end{aligned}$$

Solving for intersections,

$$\begin{aligned} & (x+a)^2 + b = (x+c)^2 + d \\ \Longrightarrow\ & (2a-2c)x = c^2 + d - a^2 - b \\ \Longrightarrow\ & x = \frac{c^2 + d - a^2 - b}{2a - 2c}. \end{aligned}$$

The only way in which this linear equation can have no roots is if $2a - 2c = 0$. Hence, $a = c$, which places the vertices on the same line $x = -a = -c$. So, the parabolae are translations of one another parallel to the y axis. □

2802. According to the best-fit model,

$$D = 1.523e^{2.049M}.$$

We also know that $M = \ln R$. Substituting for M,

$$\begin{aligned} D &= 1.523e^{2.049 \times \ln R} \\ &\equiv 1.523e^{\ln R^{2.049}} \\ &\equiv 1.523R^{2.049}. \end{aligned}$$

This is inconsistent with the assumption that D and R are related linearly. The relationship is close to quadratic, with $D \approx kR^2$.

2803. This is a quadratic in $x\sqrt{y}$:

$$\begin{aligned} & x^2\sqrt{y} + x = \frac{20}{\sqrt{y}} \\ \Longrightarrow\ & x^2 y + x\sqrt{y} - 20 = 0 \\ \Longrightarrow\ & \big(x\sqrt{y} + 5\big)\big(x\sqrt{y} - 4\big) = 0 \\ \Longrightarrow\ & \sqrt{y} = -\tfrac{5}{x}, \tfrac{4}{x}. \end{aligned}$$

Because $\sqrt{y}$ cannot be negative, these two options give real roots depending on the sign of x. For negative x, the former is viable; for positive x, the latter. Squaring the relevant results,

$$y = \begin{cases} \dfrac{25}{x^2}, & x < 0, \\ \dfrac{16}{x^2}, & x > 0. \end{cases}$$

2804. (a) There is only one order of SSSS. This gives

$$\begin{aligned} \mathbb{P}(\text{SSSS}) &= \tfrac{52}{52} \times \tfrac{12}{51} \times \tfrac{11}{50} \times \tfrac{9}{49} \\ &= \tfrac{44}{4165}. \end{aligned}$$

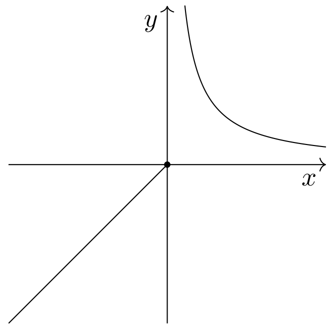
(b) There are ${}^4\text{C}_2 = 6$ orders of BBRR. This gives

$$\begin{aligned} \mathbb{P}(\text{BBRR}) &= 6 \times \tfrac{26}{52} \times \tfrac{25}{51} \times \tfrac{26}{50} \times \tfrac{25}{49} \\ &= \tfrac{325}{833}. \end{aligned}$$

2805. A counterexample is the discontinuous function

$$\text{f}(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ x, & x \leq 0. \end{cases}$$

This is one-to-one over $\mathbb{R}$ and hence invertible, but is increasing for $x < 0$ and decreasing for $x > 0$:



2806. (a) The x intercept is at $ax^2 - 1 = 0$, so $x = \frac{1}{\sqrt{a}}$. The areas are equal, so the signed areas are negatives:

$$\begin{aligned} & -\int_0^{\frac{1}{\sqrt{a}}} ax^2 - 1\,dx = \int_{\frac{1}{\sqrt{a}}}^3 ax^2 - 1\,dx \\ \Longrightarrow\ & -\Big[\tfrac{1}{3}ax^3 - x\Big]_0^{\frac{1}{\sqrt{a}}} = \Big[\tfrac{1}{3}ax^3 - x\Big]_{\frac{1}{\sqrt{a}}}^3 \\ \Longrightarrow\ & \tfrac{2}{3\sqrt{a}} = 9a - 3 + \tfrac{2}{3\sqrt{a}} \\ \Longrightarrow\ & a = \tfrac{1}{3}. \end{aligned}$$

(b) Substituting $a = \frac{1}{3}$, the area shaded is

$$2 \times \tfrac{2}{3\sqrt{a}} = \tfrac{4}{3\sqrt{3}}.$$

(c) The signed areas cancel, giving 0.

2807. For SPs, we set the first derivative to zero:

$$\begin{aligned} & -2x^{-\frac{3}{2}} + \tfrac{1}{2}x^{-\frac{1}{2}} = 0 \\ \Longrightarrow\ & -2 + \tfrac{1}{2}x = 0 \\ \Longrightarrow\ & x = 4. \end{aligned}$$

There is a stationary point at $(4, 4)$. The second derivative is

$$\frac{d^2y}{dx^2} = 3x^{-\frac{5}{2}} - \tfrac{1}{4}x^{-\frac{3}{2}}.$$

Evaluating this at $x = 4$, we get $\frac{1}{16}$. This is +ve, so the stationary point is a local minimum.

2808. (a) $A \setminus \big(B \setminus A\big) = A$

(b) $A \setminus \big(B \setminus A'\big) = A \cap B'$

(c) $A \setminus \big(B' \setminus A'\big) = A \cap B$.

2809. If a, b, c, d is an AP, then it must be symmetrical about its mean, which is both $\frac{a+d}{2}$ and $\frac{b+c}{2}$. So, these must be the same. This tells us that the mean fact is necessary.

But it isn't sufficient. The sequence $1, 4, 6, 9$ is a counterexample: the mean of 1 and 9 is 5, as is the mean of 4 and 6. But $1, 4, 6, 9$ is not an AP.

— Nota Bene —

The terms "necessary" and "sufficient", when used mathematically, are yet another way of encoding implications, to go with if-then and arrows.

A forwards implication can be expressed as

- If A then B,
- $A \implies B$,
- A is sufficient for B.

The implication in the opposite direction is

- If B then A,
- $A \impliedby B$,
- A is necessary for B.

And the two-way implication is

- A if and only if B,
- $A \iff B$,
- A is both necessary and sufficient for B.

2810. The points that satisfy $x^2 > y^3$ are on one side of the curve $x^2 = y^3$. So, if this curve crosses the circle $x^2 + (y-2)^2 = 2$, then there must be points satisfying both relations. Substituting for x^2 in the circle equation,

$$\begin{aligned} & y^3 + (y-2)^2 = 2 \\ \implies & y^3 + y^2 - 4y + 2 = 0 \\ \implies & (y-1)(y^2 + 2y - 2) = 0 \\ \implies & y = 1, -1 \pm \sqrt{3}. \end{aligned}$$

Substituting $y = 1$ gives $x = \pm 1$, so there are at least two points where the curves cross; $y = 1$ is a single root, so these cannot be points of tangency. Therefore, there are (x, y) points which satisfy the relations.

2811. The statement doesn't hold.

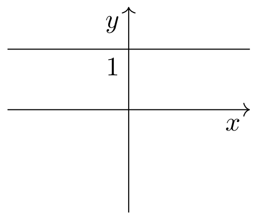
The form of the alternative hypothesis tells us that this is a two-tailed test, which means the critical region is split between the two tails. At the 1% significance level, therefore, we should be looking to see whether $\mathbb{P}(X \leq k) < 0.005$.

2812. Writing over base e, the LHS is

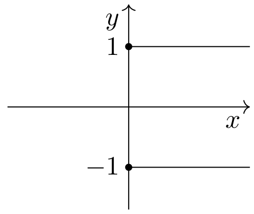
$$a^{\ln b} \equiv \left(e^{\ln a}\right)^{\ln b} \equiv e^{\ln a \ln b}.$$

Since this expression is symmetrical in a and b, it must also equal the RHS, proving the identity.

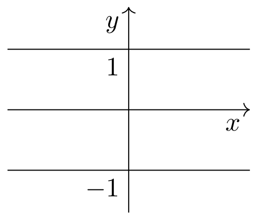
2813. (a) We reflect the negative values of $\mathrm{f}(x)$:



(b) We include extra negative values of y wherever there are positive values of y in the original graph. Where $\mathrm{f}(x) < 0$, there are no points.



(c) For any $x \in \mathbb{R}$, both of the values $y = \pm 1$ satisfy the equation:



2814. The vectors are perpendicular if $m_1 m_2 = -1$. So, we attempt to solve

$$\begin{aligned} & \frac{x+2}{x} \times \frac{x+3}{x-1} = -1 \\ \implies & (x+2)(x+3) = -x(x-1) \\ \implies & x^2 + 2x + 3 = 0. \end{aligned}$$

This has discriminant $\Delta = -8 < 0$, so it has no real roots. Hence, the vectors $\mathbf{p}$ and $\mathbf{q}$ can never be perpendicular.

2815. The probability p that all five values are below the upper quartile is

$$\begin{aligned} p &= \tfrac{75}{100} \times \tfrac{74}{99} \times \tfrac{73}{98} \times \tfrac{72}{97} \times \tfrac{71}{96} \\ &= 0.22924.... \end{aligned}$$

Hence, the probability that at least one is at or above the upper quartile is

$$1 - 0.22924... = 0.771 \text{ (3sf)}.$$

2816. (a) Rearranging, we have $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$. Squaring and adding, the first Pythagorean trig identity gives

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

(b) The components of velocity are

$$\frac{dx}{dt} = -3\sin t,$$
$$\frac{dy}{dt} = 2\cos t.$$

These have ranges $[-3, 3]$ and $[-2, 2]$. Hence, the speed is largest when the x speed is largest. This is when $\sin t = \pm 1$ and $\cos t = 0$. This gives the distance from the origin as 2, which is minimal.

— Alternative Method —

The square of the speed is

$$\begin{aligned} v^2 &= (-3\sin t)^2 + (2\cos t)^2 \\ &\equiv 9\sin^2 t + 4\cos^2 t \\ &\equiv 5\sin^2 t + 4\sin^2 t + 4\cos^2 t \\ &\equiv 5\sin^2 t + 4. \end{aligned}$$

So, the (squared) speed is maximised when $|\sin t|$ is maximised. This occurs when $|\cos t|$, and therefore the distance from the origin, is minimised.

2817. Assume, for a contradiction, that, for constants k_1, k_2, k_3, we have

$$\begin{aligned} a &= k_1 b^2, \\ b &= k_2 c^2, \\ c &= k_3 a^2. \end{aligned}$$

Substituting the third equation into the second, and the second into the first,

$$\begin{aligned} & a = k_1\Big(k_2\big(k_3 a^2\big)^2\Big)^2 \\ \implies & a - \big(k_1 k_2^2 k_3^4\big)a^6 = 0 \\ \implies & a\Big(1 - \big(k_1 k_2^2 k_3^4\big)a^5\Big) = 0 \\ \implies & a = 0, \big(k_1 k_2^2 k_3^4\big)^{-\frac{1}{5}}. \end{aligned}$$

Both of these are constants, but we are told that a, b, c are non-constant. This is a contradiction. So, it is not possible for the variables to be related in this way. □

2818. (a) This is true. The combined mean is the weighted average of the individual means; this must necessarily lie between the two.

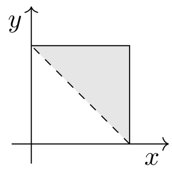
(b) This is not necessarily true. If the two sets have means that differ greatly compared to their standard deviations (e.g. heights of fleas and heights of elephants), then the standard deviation of the combined sample will be much larger than the standard deviation of either.

2819. Completing the square, the first graph is

$$\begin{aligned} y &= x^2 - 2x \\ &\equiv (x-1)^2 - 1. \end{aligned}$$

The vertices have the same y coordinate $y = -1$. Also, the graphs are both monic parabolae, so the transformation is a translation in the x direction. The vector is $-\mathbf{i}$.

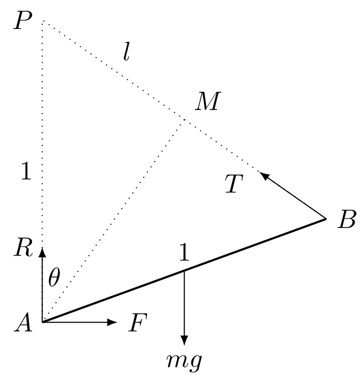
2820. Consider the possibility space as a unit square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.



The successful outcomes have $x + y > 1$, which is the shaded region. This is half of the possibility space, so the probability is $\frac{1}{2}$.

2821. The amplitude of $a\cos x + b\sin x$ is $\sqrt{a^2 + b^2}$. So, the amplitude of $6\cos x + 7\sin x$ is $\sqrt{6^2 + 7^2} \approx 9.2$. Since this is greater than 9, there are values of $6\cos x + 7\sin x + 9$ which are negative. And raising to the power $\frac{3}{4}$ requires calculation of the fourth root, which is not well defined for negative inputs. Hence, $\mathrm{f}(x)$ is not well defined on $\mathbb{R}$.

2822. (a) Let M be the midpoint of BP. Then the force diagram for the rod is as follows:



(b) Because $\triangle ABP$ is isosceles, we can split BP in half, giving $\sin\theta = l$. We take moments about A. The moment of the tension is

$$T \times |AM| = T\cos\theta.$$

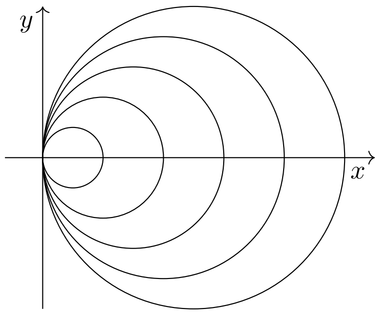
The angle of inclination of the rod is $90° - 2\theta$, so the moment of the weight is

$$\begin{aligned} & mg \times \tfrac{1}{2}\cos(90° - 2\theta) \\ \equiv\ & \tfrac{1}{2} mg \sin 2\theta \\ \equiv\ & mg\sin\theta\cos\theta. \end{aligned}$$

Equating the two moments,

$$\begin{aligned} T\cos\theta &= mg\sin\theta\cos\theta \\ \Longrightarrow T &= mg\sin\theta \\ &= mgl, \text{ as required.} \end{aligned}$$

2823. Such an AP is $a, 2a, 3a, ...$ which we can sketch as the natural numbers $n = 1, 2, 3, ...$ So, we have a family of circles of radius n, centred on $(n, 0)$. Each circle passes through the origin:



2824. The arc length, with θ defined in radians, is

$$l = r\theta.$$

Differentiating this wrt time (r constant),

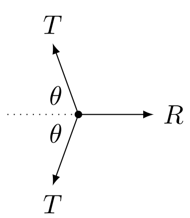
$$\frac{dl}{dt} = r\frac{d\theta}{dt}.$$

This quantity $\frac{dl}{dt}$ is the rate at which arc length i.e. distance is covered. It is therefore the speed.

2825. To reflect in the x axis, we replace y by $-y$. Hence, the new equation is $\mathrm{f}(x) + \mathrm{g}(-y) = 1$.

2826. We rearrange to $x = \operatorname{cosec} t$, $y = \cot t$. The third Pythagorean trig identity is $\operatorname{cosec}^2 t + 1 \equiv \cot^2 t$, which gives $\cot^2 t - \operatorname{cosec}^2 t \equiv 1$. Substituting into this, the Cartesian equation is $y^2 - x^2 = 1$.

2827. Each interior angle is $\pi - \frac{2\pi}{n}$, of which half is $\theta = \frac{\pi}{2} - \frac{\pi}{n}$. For a peg, the force diagram is



Resolving along the dotted line, the force exerted by the string is

$$\begin{aligned} F &= 2T\cos\theta \\ &= 2T\cos\left(\tfrac{\pi}{2} - \tfrac{\pi}{n}\right) \\ &\equiv 2T\sin\tfrac{\pi}{n}, \text{ as required.} \end{aligned}$$

2828. Differentiating both sides,

$$\begin{aligned} \int y\,dx &= 3y + c \\ \Longrightarrow y &= 3\frac{dy}{dx}. \end{aligned}$$

This is a separable DE. Separating the variables and integrating,

$$\begin{aligned} \int 3\,dx &= \int \frac{1}{y}\,dy \\ \Longrightarrow 3x + c &= \ln|y| \\ \Longrightarrow y &= Ae^{3x}. \end{aligned}$$

— Nota Bene —

The constant A can take any real value.

2829. This is not true. Applying f to both sides of the equation gives $\mathrm{f}^2(k) = k$. This means k is fixed under the action of f^2, but not necessarily under f. The reciprocal function $\mathrm{g}(x) = \frac{1}{x}$, which is self-inverse, is a counterexample: $\mathrm{g}(2) = \mathrm{g}^{-1}(2) = \frac{1}{2}$. But 2 is not a fixed point of the reciprocal function. This disproves the statement.

2830. (a) There are roots at $x = a, b, c$. A polynomial curve cannot depart from $y = 0$ and return to it without turning in between. Such turning points are stationary points.

(b) Let $x = k$ be the SP in (b, c). Then the first derivative is zero at both $x = k$ and $x = c$, since a double root is an SP. Hence, by the same argument as in (a), the first derivative must turn somewhere in (k, c), which is a subset of (b, c). A point where the first derivative turns is a point of inflection.

There may or may not be a point of inflection in (a, b). The curve $y = x(x-1)(x-10)^2$ is a counterexample: it satisfies the conditions of the question with $a = 0$, $b = 1$, $c = 10$, but both of its points of inflection are in $(1, 10)$.

2831. We can take $(1 + x + x^2)$ out of the sum, since it is independent of the index i:

$$(1 + x + x^2)\sum_{i=1}^{\infty} x^i = \tfrac{13}{18}.$$

This leaves a geometric series with first term x and common ratio x. Using $S_\infty = \frac{a}{1-r}$, its infinite sum (valid for $|x| < 1$) is $\frac{x}{1-x}$:

$$\begin{aligned} \left(1 + x + x^2\right)\tfrac{x}{1-x} &= \tfrac{13}{18} \\ 18x + 18x^2 + 18x^3 &= 13 - 13x \\ \Longrightarrow 18x^3 + 18x^2 + 31x - 13 &= 0. \end{aligned}$$

Using a polynomial solver, $x = 1/3$. Since $1/3 < 1$, the geometric sum to infinity converges. Hence, $x = 1/3$ satisfies the equation.

2832. Let the people be

$$A_1, A_2, B_1, B_2, C_1, C_2.$$

Place A_1 wlog. Then the probability that A_2 sits opposite is $\frac{1}{5}$. Place B_1 wlog. The probability that B_2 sits opposite is $\frac{1}{3}$. C_1 and C_2 are automatically opposite. So,

$$p = \tfrac{1}{5} \times \tfrac{1}{3} = \tfrac{1}{15}.$$

——— Alternative Method ———

For a combinatorial approach, there are 6! ways in which the three couples can sit. Of these, there are 6 ways for the first couple to sit opposite one another, leaving 4 ways for the second couple and 2 ways for the third couple. This gives

$$p = \frac{6 \times 4 \times 2}{6!} = \tfrac{1}{15}.$$

2833. Consider the zero function $\mathrm{f}(x) = 0$ for all $x \in \mathbb{R}$, and the "almost-zero" function

$$\mathrm{g} : \begin{cases} x \mapsto 0, & x \neq 0, \\ x \mapsto 1, & x = 0. \end{cases}$$

These functions are not identical, but, since the area of the (x, y) plane enclosed below the point $(0, 1)$ is necessarily zero, their definite integrals are the same: they are both identically zero. This is a counterexample and disproves the statement.

2834. Differentiating, $\frac{dy}{dx} = 2ax + b$. The tangents are

$$\begin{aligned} &y = (2aq + b)x + k_1, \text{ through } (q, aq^2 + bq + c), \\ &y = (-2aq + b)x + k_2, \text{ through } (-q, aq^2 - bq + c). \end{aligned}$$

Substituting to find the constants k_1 and k_2,

$$\begin{aligned} aq^2 + bq + c &= (2aq + b)q + k_1, \\ aq^2 - bq + c &= (-2aq + b) \cdot -q + k_2. \end{aligned}$$

Each of these simplifies to the same value:

$$k_1 = k_2 = -aq^2 + c.$$

Therefore, since the tangent lines have the same y intercept, they must cross on the y axis. □

2835. Without loss of generality, we can scale and rotate the scenario such that $\mathbf{p} = \mathbf{i}$ and $\mathbf{q} = k\mathbf{j}$. So, we know that $2\mathbf{i}+3k\mathbf{j}$ and $-3\mathbf{i}+2k\mathbf{j}$ are perpendicular. Algebraically, this is

$$\begin{aligned} &\frac{3k}{2} \times \frac{2k}{-3} = -1 \\ \Longrightarrow\ &k^2 = 1 \\ \Longrightarrow\ &k = \pm 1. \end{aligned}$$

Hence, $\mathbf{p} = \pm\mathbf{q}$, so $|\mathbf{p}| = |\mathbf{q}|$.

2836. The first derivative h' has a double root at $x = 1$, meaning that $\mathrm{h}'(x) = k(x-1)^2$, for some constant k. Integrating this,

$$\mathrm{h}(x) = \tfrac{1}{3}k(x-1)^3 + c.$$

Substituting the two known points, $-\frac{1}{3}k + c = -1$ and $\frac{1}{3}k + c = 7$. Solving simultaneously, $k = 12$ and $c = 3$. The curve is stationary at $x = 1$, since $\mathrm{h}'(1) = 0$. So, the coordinates of the stationary point are $(1, 3)$.

2837. We start with the LHS, multiplying top and bottom by $(1 + \sin\theta)$:

$$\begin{aligned} &\frac{1+\sin\theta}{1-\sin\theta} \\ \equiv\ &\frac{(1+\sin\theta)^2}{1-\sin^2\theta} \\ \equiv\ &\frac{(1+\sin\theta)^2}{\cos^2\theta} \\ \equiv\ &\left(\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right)^2 \\ \equiv\ &(\sec\theta + \tan\theta)^2, \text{ as required.} \end{aligned}$$

2838. The relevant definitions are

$$\begin{aligned} \text{Even} &: \mathrm{f}(-x) \equiv \mathrm{f}(x), \\ \text{Odd} &: \mathrm{f}(-x) \equiv -\mathrm{f}(x). \end{aligned}$$

If both of these are true, then $-\mathrm{f}(x) \equiv \mathrm{f}(x)$, which implies that $2\,\mathrm{f}(x) \equiv 0$. Hence f must be the zero function. □

2839. Integrating the velocities, the displacements at time T are

$$\begin{aligned} \int_0^T \frac{1}{1+t}\,dt &\equiv \Big[\ln|1+t|\Big]_0^T \\ &\equiv \ln(1+T), \\ \int_0^T \frac{1}{1+2t}\,dt &\equiv \Big[\tfrac{1}{2}\ln|1+2t|\Big]_0^T \\ &\equiv \tfrac{1}{2}\ln(1+2T). \end{aligned}$$

So, we solve $\ln(1+T) = {}^1\!/_2 \ln(1+2T)$. The RHS may be rewritten as $\ln\sqrt{1+2T}$. Exponentiating both sides,

$$\begin{aligned} &1 + T = \sqrt{1+2T} \\ \Longrightarrow\ &1 + 2T + T^2 = 1 + 2T \\ \Longrightarrow\ &T^2 = 0. \end{aligned}$$

Hence, the displacements are only equal at $T = 0$, and not thereafter.

2840. Taking logs of the first equation,

$$\begin{aligned} & xy = a^2 \\ \implies & \log_a(xy) = 2 \\ \implies & \log_a x + \log_a y = 2. \end{aligned}$$

The second equation is a difference of two squares, so we can factorise and then substitute:

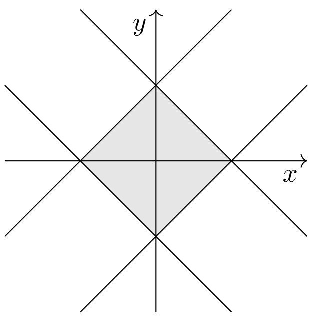
$$\begin{aligned} & (\log_a x)^2 - (\log_a y)^2 = 1 \\ \implies & (\log_a x + \log_a y)(\log_a x - \log_a y) = 1 \\ \implies & 2(\log_a x - \log_a y) = 1 \\ \implies & \log_a x - \log_a y = \tfrac{1}{2}. \end{aligned}$$

This gives a pair of linear simultaneous equations in $\log_a x$ and $\log_a y$, which we can solve to give $\log_a x = \frac{5}{4}$ and $\log_a y = \frac{3}{4}$. Hence, the solution is $x = a^{\frac{5}{4}}$ and $y = a^{\frac{3}{4}}$.

2841. (a) No, this would contain $(0, 1)$.
(b) No, likewise.
(c) Yes.
(d) No, this would be tangent to the x axis, rather than crossing it.

2842. The first derivative is $12x^2 + 7$. Looking for SPs, we set $12x^2 + 7 = 0$, which has no real roots. So, there are no SPs. Hence, the cubic $y = 4x^3 + 7x - 8$ is increasing everywhere. This means it must cross the x axis exactly once, as required.

2843. The boundary equations are $x + y = \pm 1$ and $x - y = \pm 1$. These are two pairs of parallel lines, intersecting at the vertices of a square. The region satisfying both inequalities is shaded:



2844. The variance is the average squared deviation from the mean, which is the average of the summands in (b). Hence, the average value of the summands in (b) is 7.281.

The summands in (a) are the (positive) square roots of these quantities, which, since $7.281 > 1$, must, on average, be smaller than the expressions in (b).

So, the sum in (b) is guaranteed to be larger than the sum in (a).

2845. Solving for intersections yield a non-analytically solvable equation $2x - e = x \ln x$. So, instead, we differentiate by the product rule:

$$\frac{dy}{dx} = \ln x + 1.$$

Equating this to 2, we get $\ln x = 1$, which gives $x = e$. At the coordinates (e, e), the tangent line is $y - e = 2(x - e)$. This simplifies to $y = 2x - e$, as required.

2846. There are 7! different ways of rearranging a set of 7 objects. But the three As are indistinguishable, so we overcount each arrangement by 3! orders of those. Hence, the total number of arrangements is $\frac{7!}{3!} = 840$. So, the probability is $\frac{1}{840}$.

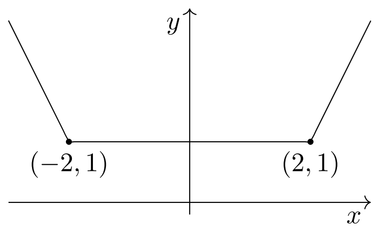
— Alternative Method —

Using a conditioning approach, we select the first letter, then the second and so on:

$$p = \tfrac{3}{7} \times \tfrac{1}{6} \times \tfrac{2}{5} \times \tfrac{1}{4} \times \tfrac{1}{3} \times \tfrac{1}{2} = \tfrac{1}{840}.$$

2847. (a) The vertices are at the points with zero inputs to the mod functions. This gives $(-2, 1)$ and $(2, 1)$.
(b) If the input to a mod function is positive, then the mod function is *passive*. If the input is negative, however, then the mod function is *active*, applying a negative sign to the input:
- Over the domain $(-\infty, -2)$, both mod functions are active, so the gradient is -2.
- Over the domain $(-2, 2)$, one is active and one passive, so the gradient is 0.
- Over the domain $(2, \infty)$, both are passive, so the gradient is 2.

(c) Collating the above information, G is



2848. Assume the cube has unit side length. By 2D and 3D Pythagoras respectively, the face diagonal has length $\sqrt{2}$ and the space diagonal has length $\sqrt{3}$. These are the adjacent and hypotenuse of a right-angled triangle, giving the angle as $\arccos \sqrt{2}/\sqrt{3}$.

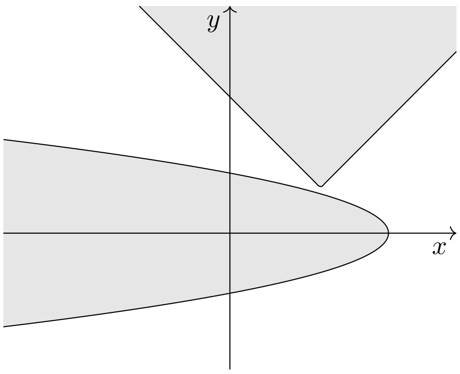
2849. The derivative of $\cot x$ is $-\operatorname{cosec}^2 x$. Subbing this into the LHS gives $\cot^2 x - \operatorname{cosec}^2 x$, which we can rewrite as $-(\operatorname{cosec}^2 x - \cot^2 x)$. According to the third Pythagorean trig identity, the expression in the brackets is 1, as required.

2850. Assume, for a contradiction, that neither SP is a point of inflection. Then both are turning points, which means the gradient has the same sign as $x \to \pm\infty$.

Assume instead, for another contradiction, that both SPs are points of inflection. Then neither is a turning point, which means that the gradient, as before, has the same sign as $x \to \pm\infty$.

Since $y = \mathrm{g}(x)$ has even degree, its gradients must have opposite signs as $x \to \pm\infty$. In both cases above, this is a contradiction. Hence, exactly one of the SPs must be a point of inflection. □

2851. The boundary equations are $y = 2 + |x - 4|$ and $x = 7 - y^2$. The former has a vertex at $(4, 2)$. This lies above the parabola, which is at $\left(4, \sqrt{3}\right)$, with gradient $m = -1/2\sqrt{3}$. Since $|m| < 1$, the parabola and the mod graph do not intersect. Hence, there are no (x, y) points which are simultaneously above the mod graph and to the left of the parabola:



2852. We know that, as $\theta \to 0$, $\cos\theta \to 1$. And $\sec\theta$ is the reciprocal of $\cos\theta$, so it must also tend to 1. According to the given inequality, this sandwiches $\frac{\theta}{\sin\theta}$ between 1 and $\sec\theta$, which is tending to 1. Hence,

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1.$$

The ratio between the two quantities θ and $\sin\theta$ approaches 1, so $\sin\theta \approx \theta$ for small θ.

——— Nota Bene ———

This idea of trapping one quantity between two others is known as the *squeeze theorem.*

2853. (a) This statement doesn't hold: $a = 2.5$, $b = -2$ is a counterexample.

(b) This doesn't hold either: $a = b = -0.5$ is a counterexample.

2854. Solving for intersections, we have

$$\begin{aligned} & x + (x^2 + x) = \left(x - (x^2 + x)\right)^2 + 2 \\ \Longrightarrow\ & x^2 + 2x = x^4 + 2 \\ \Longrightarrow\ & x^4 - x^2 - 2x + 2 = 0. \end{aligned}$$

Using a polynomial solver, $x = 1$. This gives

$$(x - 1)(x^3 + x^2 - 2) = 0.$$

Using a polynomial solver on the cubic, we again get $x = 1$. Hence, $x = 1$ is a double root of the original equation, meaning that the intersection is a point of tangency. Explicitly,

$$\begin{aligned} & x^4 - x^2 - 2x + 2 = 0 \\ \Longleftrightarrow\ & (x - 1)^2(x^2 + 2x + 2) = 0. \end{aligned}$$

2855. There are now two separate systems.

Considering the left-hand blocks as one system, we resolve along the string:

$$\begin{aligned} & mg - mg\sin 30° = 2ma_1 \\ \Longrightarrow\ & a_1 = \tfrac{1}{4}g. \end{aligned}$$

The right-hand block accelerates at $a_2 = \frac{1}{2}g$. Hence, the magnitude of the acceleration of each slope block, relative to the other, is $a_1 + a_2 = \frac{3}{4}g$. We can now set up a *suvat* for this gap:

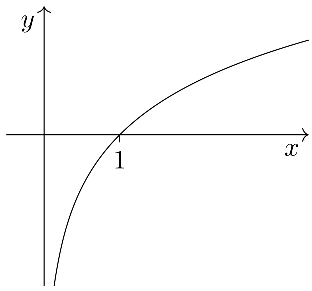
$$\begin{aligned} & 0.50 = \tfrac{1}{2} \times \tfrac{3}{4}gt^2 \\ \Longrightarrow\ & t = \pm\tfrac{2}{\sqrt{3g}}. \end{aligned}$$

Since $t > 0$, the time taken is 0.369 s (3sf).

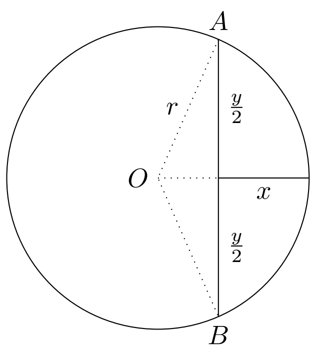
2856. We can raise base and input to the same power:

$$\log_2 x \equiv \log_4 x^2 \equiv 2\log_4 x.$$

Hence, the graph simplifies to $y = \log_4 x$, which is a standard log graph:



2857. (a) The scenario is



(b) Using Pythagoras,

$$\begin{aligned} &(r-x)^2 + \left(\frac{y}{2}\right)^2 = r^2 \\ \Longrightarrow\ & r^2 - 2rx + x^2 + \frac{y^2}{4} = r^2 \\ \Longrightarrow\ & 2rx = x^2 + \frac{y^2}{4} \\ \Longrightarrow\ & r = \frac{y^2}{8x} + \frac{x}{2}, \text{ as required.} \end{aligned}$$

2858. Using $\cos 2\theta \equiv 2\cos^2\theta - 1$,

$$\begin{aligned} &\cos^2\theta = \cos 2\theta + 1 \\ \Longrightarrow\ & \cos^2\theta = 2\cos^2\theta \\ \Longrightarrow\ & \cos\theta = 0 \\ \Longrightarrow\ & \theta = \pm\tfrac{\pi}{2}. \end{aligned}$$

2859. Assume, for a contradiction, that $g(x)$ has even degree. So, $g'(x)$ has odd degree, meaning that $g'(x) = 0$ must have at least one real root. So, $g(x)$ must have at least one SP.

The function g is increasing everywhere. Hence, its first derivative is positive everywhere. So, the curve has no SPs. This is a contradiction. Hence, g must be of odd degree. □

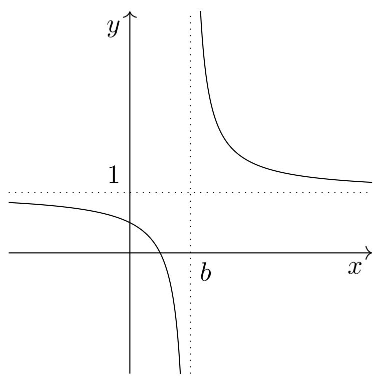
2860. Since $a < b$, we know that $x-a > x-b$. Therefore, if both numerator and denominator are positive, then the fraction will be greater than 1. If, on the other hand, the denominator is negative, then the fraction will be less than 1. So, the solution set is (b, ∞).

— Alternative Method —

The boundary equation is $x - a = x - b$, which has no roots. Consider the curve

$$y = \frac{x-a}{x-b} \equiv \frac{x-b+(b-a)}{x-b} \equiv 1 + \frac{b-a}{x-b}.$$

Since $b - a > 0$, this is a positive reciprocal graph. It has asymptotes at $x = b$ and $y = 1$:



The curve lies above $y = 1$ for $x \in (b, \infty)$. So, this is the solution of the inequality.

2861. To translate 4 units in the positive y direction, we replace y by $y - 4$, giving $f(x) + f(y-4) = k$.

2862. (a) The squared distance is given by Pythagoras:

$$\begin{aligned} d^2 &= \big((2t+1) - (1-t)\big)^2 + \big((1) - (4t)\big)^2 \\ &\equiv 25t^2 - 8t + 1. \end{aligned}$$

(b) Equating the above to 18, we find $t = -17/25, 1$. The squared distance is a positive quadratic, so the particles are closer together than $\sqrt{18}$ for $t \in (-17/25, 1)$.

2863. The best linear approximation is the tangent line. Finding the first derivative,

$$\begin{aligned} &f'(x) = 2x\cos(x^2), \\ \Longrightarrow\ & f'(\sqrt{\pi}) = 2\sqrt{\pi}\cos\pi = -2\sqrt{\pi}. \end{aligned}$$

We also know that $f(\sqrt{\pi}) = \sin\pi = 0$. Using a functional version of $y - y_1 = m(x - x_1)$,

$$\begin{aligned} &g(x) - f(\sqrt{\pi}) = -2\sqrt{\pi}(x - \sqrt{\pi}) \\ \Longrightarrow\ & g(x) = 2\pi - 2\sqrt{\pi}x, \text{ as required.} \end{aligned}$$

2864. (a) $4^x + 15^x - 3^x - 20^x \equiv (1 - 5^x)(4^x - 3^x)$.

(b) For this expression to be zero, one of its factors must be zero. So, $5^x = 1 \implies x = 0$, or else

$$\begin{aligned} &4^x = 3^x \\ \Longrightarrow\ & (4/3)^x = 1 \\ \Longrightarrow\ & x = 0. \end{aligned}$$

Hence, the expression can only be zero when $x = 0$.

2865. Splitting the fraction up, the integrand is $x - \frac{1}{x^2}$:

$$\begin{aligned} &\int_a^{2a} x - \tfrac{1}{x^2}\,dx \\ \equiv\ & \left[\tfrac{1}{2}x^2 + \tfrac{1}{x}\right]_a^{2a} \\ \equiv\ & \left(\tfrac{1}{2}(2a)^2 + \tfrac{1}{2a}\right) - \left(\tfrac{1}{2}(a)^2 + \tfrac{1}{a}\right) \\ \equiv\ & \tfrac{3}{2}a^2 - \tfrac{1}{2a} \\ \equiv\ & \frac{3a^3 - 1}{2a}, \text{ as required.} \end{aligned}$$

2866. Assume, for a contradiction, that two of the lines of action intersect at P and the other doesn't pass through P.

Consider moments about P. The first two forces have zero moment, while the third has a non-zero moment. Hence, the resultant moment is non-zero, and the object cannot be in equilibrium. This is a contradiction. So, for equilibrium, the three forces must have concurrent lines of action. □

2867. (a) The derivative is $\frac{dy}{dx} = 15x^2 - 6$. At $x = -4$, this is 234. So, at the point $(-4, -288)$, the equation of the tangent is

$$\begin{aligned} y + 288 &= 234(x+4) \\ \Longrightarrow y &= 234x + 648, \text{ as required.} \end{aligned}$$

(b) Solving simultaneously for intersections,

$$\begin{aligned} 5x^3 - 6x + 8 &= 234x + 648 \\ \Longrightarrow x^3 - 48x - 128 &= 0. \end{aligned}$$

We know that this has a factor of $(x+4)^2$, from the point of tangency. Taking it out, we have $(x+4)^2(x-8) = 0$. So, T re-intersects the curve at $x = 8$.

2868. This is a quadratic in $\ln x$:

$$\begin{aligned} 5\ln x + 4 &= \frac{1}{\ln x} \\ \Longrightarrow 5(\ln x)^2 + 4\ln x - 1 &= 0 \\ \Longrightarrow (5\ln x - 1)(\ln x + 1) &= 0 \\ \Longrightarrow \ln x &= \tfrac{1}{5}, -1 \\ \Longrightarrow x &= \sqrt[5]{e}, 1/e. \end{aligned}$$

2869. Setting $y = 0$, we have t limits of ± 1. Using the parametric integration formula,

$$\begin{aligned} A &= \int_{t_1}^{t_2} y \frac{dx}{dt}\, dt \\ &= \int_{-1}^{1} (1 - t^2) 3t^2\, dt \\ &= \int_{-1}^{1} 3t^2 - 3t^4\, dt. \end{aligned}$$

Carrying out the integral, this is

$$\begin{aligned} &\left[t^3 - \tfrac{3}{5}t^5\right]_{-1}^{1} \\ &= \left(1 - \tfrac{3}{5}\right) - \left(-1 + \tfrac{3}{5}\right) \\ &= \tfrac{4}{5}. \end{aligned}$$

2870. Assume, for a contradiction, that the polynomial f is periodic, with period T.

So, $f(nT) = f(0)$ for all $n \in \mathbb{Z}$. This means that the equation $f(x) = f(0)$ has infinitely many roots. But $f(x) = f(0)$ is a polynomial equation, and has finitely many roots. This is a contradiction. Hence, no polynomial function is periodic. □

— Nota Bene —

Since a polynomial of degree n can have at most n linear factors of the form $x - k$, the factor theorem shows that a polynomial equation of degree n can have at most n roots.

2871. (a) This is true. Having excluded 0, a quotient of two positive numbers is a positive number.

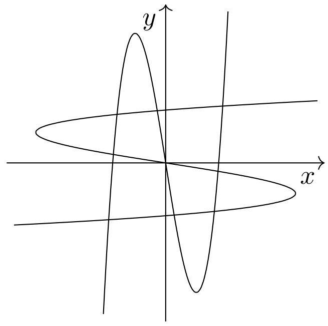
(b) This is true, for the same reason as in (a).

(c) This is false: $x = 1$, $y = 0$ is a counterexample, as $1/0$ is undefined.

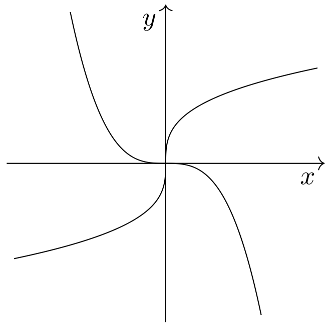
— Nota Bene —

Division by zero isn't defined as giving ∞; this is why ∞ doesn't appear in a square bracket in interval notation. Infinity isn't a number on the number line, but rather a broad concept describing growth without limit.

2872. (a) The greatest value of n is 9:



(b) The least value of n is 1:



2873. Factorising,

$$\begin{aligned} \cos^5\theta &= -\cos^3\theta \\ \Longrightarrow \cos^3\theta(\cos^2\theta + 1) &= 0 \end{aligned}$$

The quadratic factor has no roots, as $\cos^2\theta \geq 0$. So, the only roots are when $\cos\theta = 0$. This gives the solution as $\theta = -90°, 90°$.

2874. Differentiating by the quotient rule,

$$\begin{aligned} y &= \frac{x}{x^2+4} \\ \Longrightarrow \frac{dy}{dx} &= \frac{(x^2+4) - x(2x)}{(x^2+4)^2} \\ &\equiv \frac{4 - x^2}{(x^2+4)^2}. \end{aligned}$$

Setting the numerator to zero, we find SPs at $(2, 1/4)$ and $(-2, -1/4)$.

Differentiation again by the quotient rule,

$$\frac{dy^2}{dx^2} = \frac{-2x(x^2+4)^2 - 4x\big(4-x^2\big)(x^2+4)}{(x^2+4)^4}$$
$$\equiv \frac{2x^3-24x}{(x^2+4)^3}.$$

Evaluating this at $x = \pm 2$, we get $\mp 1/16$. Hence, there is a local max at $(2, 1/4)$ and a local min at $(-2, -1/4)$. This matches the behaviour shown.

Also, the numerator has a single root at $x = 0$, as shown in the graph passing through the origin.

Furthermore, the denominator is always positive and dominates the numerator for large $|x|$, so the behaviour is $y \to 0^{\pm}$ as $x \to \pm\infty$. Again, this matches the graph.

— Nota Bene —

There is no "right amount" of answer here, as with any proof. Mathematics, just like, say, philosophy or history, is a *human activity*. As such, there is and will always be a judgement call about how much depth to go into!

2875. For $m \neq 0$, this is a quadratic. So, it has two intersections with the x axis iff $\Delta > 0$. This is

$$(m-1)^2 - 16m^2 > 0.$$

The boundary equation is $15m^2 + 2m - 1 = 0$, so $m = -1/3, 1/5$. The quadratic is negative, so the solution lies between these roots.

However, we must exclude $m = 0$, for which the equation is linear, and cannot have two roots. This gives $m \in (-1/3, 1/5) \setminus \{0\}$.

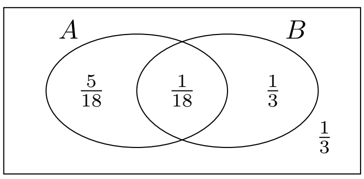
2876. We know that $x^2y^3 = a$, for some constant a. Rearranging and defining a new constant, this is $y = bx^{-\frac{2}{3}}$. Hence, $\frac{dy}{dx} = -\frac{2}{3}bx^{-\frac{5}{3}}$. Evaluating at $x = 8$ and $x = 1$, the required ratio is

$$-\tfrac{2}{3}bx^{-\frac{5}{3}}\Big|_{x=8} : -\tfrac{2}{3}bx^{-\frac{5}{3}}\Big|_{x=1}.$$

The constant $-\frac{2}{3}b$ cancels, leaving

$$8^{-\frac{5}{3}} : 1^{-\frac{5}{3}}$$
$$= \tfrac{1}{32} : 1$$
$$= 1 : 32.$$

2877. The total probability of A is $1/3$, which is $6/18$, so the uppermost branch has probability $1/18$. To find the other regions, we multiply along the lower branches:



2878. By the chain rule,

$$y = \ln\Big(x + \sqrt{1+x^2}\Big)$$
$$\Longrightarrow \frac{dy}{dx} = \frac{1 + \frac{1}{2}\big(1+x^2\big)^{-\frac{1}{2}} \cdot 2x}{x + \sqrt{1+x^2}}$$
$$\equiv \frac{1 + x\big(1+x^2\big)^{-\frac{1}{2}}}{x + \sqrt{1+x^2}}.$$

We multiply top and bottom by the conjugate of the denominator, as per rationalising surds, to give

$$\frac{dy}{dx} = \frac{\Big(1 + x\big(1+x^2\big)^{-\frac{1}{2}}\Big)\Big(x - \big(1+x^2\big)^{\frac{1}{2}}\Big)}{x^2 - (1+x^2)}$$
$$\equiv \frac{x - \big(1+x^2\big)^{\frac{1}{2}} + x^2\big(1+x^2\big)^{-\frac{1}{2}} - x}{-1}$$
$$\equiv \big(1+x^2\big)^{\frac{1}{2}} - x^2\big(1+x^2\big)^{-\frac{1}{2}}$$
$$\equiv \frac{\big(1+x^2\big) - x^2}{\big(1+x^2\big)^{\frac{1}{2}}}$$
$$\equiv \frac{1}{\sqrt{1+x^2}}, \text{ as required.}$$

2879. Call the forces A and P_1, P_2, P_3, where force A is perpendicular to the other three.

Consider the resultant force in the direction of force A. Since P_1, P_2, P_3 are all perpendicular to A, they can have no component in this direction. Hence, the resultant force in the direction of A consists only of A, which is non-zero. So, it is not possible for the object to remain in equilibrium.

2880. Using a polynomial solver on $8 = x^3 - x^2 + 4$, we get $x = 2$. The first derivative is $3x^2 - 2x$, which, at $x = 2$, has value 8. So, the tangent line is $y - 8 = 8(x - 2)$. Setting $y = 0$, we get $8x = 8$, so the tangent line crosses the x axis at $x = 1$.

2881. When we expand, we are selecting one of $1, x, x^2$ from each of the five brackets. There are two ways to get x^7, by selecting

① three instances of x^2, one of x and one of 1,
② two instances of x^2 and three of x.

There are ① ${}^5\text{C}_3 \times 2 = 20$ and ② ${}^5\text{C}_3 = 10$ ways of doing this. So, the coefficient of x^7 is 30.

2882. Neither (a) nor (c) is true. A counterexample to both is as follows. If the solution set of $\text{f}(x) \geq 0$ is $[0,\infty) \cap \{-1\}$ and the solution set of $\text{g}(x) \geq 0$ is $[0,\infty) \cap \{-2\}$, then $S = [0,\infty)$ as in the question.

The solution set of $\text{f}(x) < 0$ is $(-\infty, 0) \setminus \{-1\}$, and the solution set of $\text{g}(x) < 0$ is $(-\infty, 0) \setminus \{-2\}$. The intersection of these is $T = (-\infty, 0) \setminus \{-1, -2\}$, which does not include all of $(\infty, 0)$.

2883. Enacting the differential operator,

$$\begin{aligned} &\frac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=0 \\ \Longrightarrow\ &\tfrac{1}{2}x^{-\frac{1}{2}}+\tfrac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}=0 \\ \Longrightarrow\ &\frac{dy}{dx}=\frac{\sqrt{y}}{\sqrt{x}} \\ \Longrightarrow\ &\left(\frac{dy}{dx}\right)^2=\frac{y}{x},\text{ as required.} \end{aligned}$$

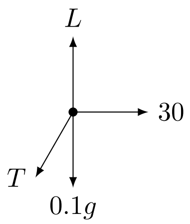
2884. The implication goes backwards. By definition and the factor theorem, $\mathrm{f}(x)$ has a double root at $x = a$ iff $\mathrm{f}(x)$ has a factor of $(x-a)^2$. This implies, but is not implied by, the presence of a factor of $(x - a)$.

2885. Using the first Pythagorean trig identity,

$$\begin{aligned} y &= \sin^2(\arccos x) \\ &\equiv 1-\cos^2(\arccos x) \\ &= 1-x^2. \end{aligned}$$

So, every point on the original curve lies on the parabola $y = 1 - x^2$. However, the domain of arccos is $[-1, 1]$, so the graph is restricted to this section of the parabola.

2886. (a) On the kite:



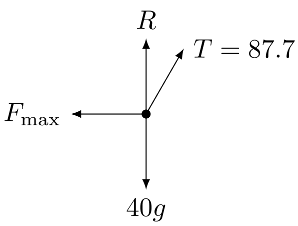
Equilibrium gives

$$\begin{aligned} \updownarrow &: L-0.1g-T\cos 20^\circ=0 \\ \leftrightarrow &: 30-T\sin 20^\circ=0. \end{aligned}$$

Solving these, $T = 30\,\text{cosec}\,20^\circ = 87.7$ N and $L = T\cos 20^\circ + 0.1g = 83.4$ N, both to 3sf.

(b) The tension in the string exerts a horizontal force on the boy. Without friction from the ground, he would slip to the right.

(c) The force diagram for the boy, when on the point of slipping to the right, is



$$\begin{aligned} \updownarrow &: R-40g+T\cos 20^\circ=0 \\ \leftrightarrow &: T\sin 20^\circ-F_{\text{max}}=0. \end{aligned}$$

So, $R = 309.576$. Then $\mu R = T\sin 20^\circ$, which gives the minimum possible value of μ to be $\mu = \frac{30}{R} = 0.0969$ (3sf).

2887. There are 30 ways of choosing the chairperson, then ${}^{29}\text{C}_2 = 406$ ways of choosing the secretaries, then ${}^{27}\text{C}_3 = 2925$ ways of choosing the adjutants. This gives $30 \times 406 \times 2925 = 35626500$ ways of choosing the officers.

2888. The even curve in (b) has the y axis as a line of symmetry, since both $y = x^6$ and $y = x^4$ have such even symmetry. But the curves in (a) and (c) have no line of symmetry: the symmetry of $y = x^3$ and $y = x^5$ is rotational rather than reflective.

(a) False,
(b) True,
(c) False.

2889. Factorising this quadratic in x^2,

$$\begin{aligned} &x^4-10k^2x^2+9k^4=0 \\ \Longrightarrow\ &(x^2-9k^2)(x^2-k^2)=0 \\ \Longrightarrow\ &x^2=k^2,9k^2 \\ \Longrightarrow\ &x=-3k,-k,k,3k. \end{aligned}$$

These roots are in arithmetic progression, with common difference $2k$.

2890. There are six ways in which this can happen:

$1,2,3,4,5$	$1,2,4,5,6$
$1,2,3,4,6$	$1,3,4,5,6$
$1,2,3,5,6$	$2,3,4,5,6.$

The probability of each is $\frac{1}{6}^5$, so the probability is

$$\mathbb{P}(\text{strictly increasing}) = 6\times\tfrac{1}{6}^5=\tfrac{1}{1296}.$$

2891. It is not possible.

With $y \geq 0$, we have only 180° available around O. Each degree can be used at most twice, giving 360° in total. But the interior angles of all four n-gons add to more than 360°:

$$60^\circ+90^\circ+108^\circ+120^\circ=378^\circ>360^\circ.$$

2892. Substituting the factorial definition of ${}^n\text{C}_r$,

$$\begin{aligned} &\frac{n!}{3!(n-3)!}+\frac{n!}{5!(n-5)!}=2\times\frac{n!}{4!(n-4)!} \\ \Longrightarrow\ &\frac{n(n-1)(n-2)}{6} \\ &\quad+\frac{n(n-1)(n-2)(n-3)(n-4)}{120} \\ &\qquad=\frac{n(n-1)(n-2)(n-3)}{12}. \end{aligned}$$

There is a common factor of $n(n-1)(n-2)$. None of these can be zero, as e.g. ${}^n\mathrm{C}_3$ is not defined for $n < 3$. So, we can divide through, leaving

$$\begin{aligned}&\frac{1}{6}+\frac{(n-3)(n-4)}{120}=\frac{n-3}{12}\\ \Longrightarrow\ & n^2-17n+62=0\\ \Longrightarrow\ & n=\tfrac{1}{2}\big(17\pm\sqrt{41}\big).\end{aligned}$$

These are non-integers. So, no integers n satisfy the equation.

2893. The first boundary equation is

$$x(x-2)(x+2)=0.$$

So, the first set is $(-2,0)\cup(2,\infty)$. And the second boundary equation is

$$(x-1)(x+1)=0.$$

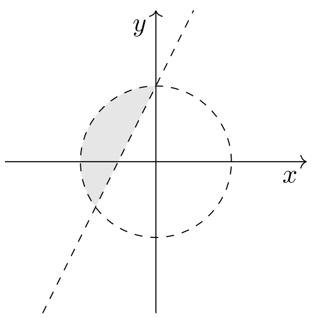
So, the second set is $[-1,1]$. The union of these is $(-2,1]\cup(2,\infty)$.

2894. The points at which this curve meets the axes are $x=0 \implies t=0$ and $y=0 \implies t=\pm 1$. Since $t=-1$ gives $(-1,0)$, the points depicted are $t_1=0$ and $t_2=1$. So, the area is given by

$$\begin{aligned}&\int_{t_1}^{t_2} y\frac{dx}{dt}\,dt\\ =&\int_0^1 (t^2-1)^2\cdot 3t^2\,dt.\end{aligned}$$

Using definite integration on a calculator, the area of the shaded region is $\frac{8}{35}$.

2895. The situation is



Solving the boundary equations simultaneously, $x^2+(2x+1)^2=1$, giving

$$A:\big(0,1\big)\quad\text{and}\quad B:\big(-\tfrac{4}{5},-\tfrac{3}{5}\big).$$

The angle subtended at the centre by the chord is $\theta=\arccos(-3/5)$. The area of the unshaded region is the area of the minor sector OAB minus the area of triangle OAB, which is

$$\begin{aligned}A&=\tfrac{1}{2}\theta-\tfrac{1}{2}\sin\theta\\ &=0.707149...\end{aligned}$$

This very close to $\frac{\sqrt{2}}{2}=0.707106...$, as required.

2896. Assume, for a contradiction, that there is such a polynomial function f, with $\mathrm{f}(x)\equiv e^x$. Then we know that $\mathrm{f}'(x)\equiv\mathrm{f}(x)$. But the derivative of a polynomial of degree n is a polynomial of degree $n-1$. This is a contradiction. So, the exponential function is not equivalent to any polynomial. □

2897. Solving simultaneously for intersections,

$$\begin{aligned}&-x=\frac{1}{1+\frac{1}{x}+\frac{1}{x^2}}\\ \Longrightarrow\ &-x=\frac{x^2}{x^2+x+1}\\ \Longrightarrow\ & x^3+2x^2+x=0\\ \Longrightarrow\ & x(x+1)^2=0.\end{aligned}$$

Since the factor $(x+1)$ is squared in the above, $x=-1$ is a double root, which means there is a point of tangency at $(1,-1)$.

2898. The right-hand equation is

$$\begin{aligned}&\tan^2\theta=k\\ \Longleftrightarrow\ &\sec^2\theta-1=k\\ \Longleftrightarrow\ &\sec\theta=\pm\sqrt{1+k}.\end{aligned}$$

Since the left-hand equation uses only the positive square root, the first implication holds, but the second doesn't.

2899. The t derivatives are

$$\begin{aligned}\frac{dx}{dt}&=1-\cos t,\\ \frac{dy}{dt}&=\sin t.\end{aligned}$$

By the chain rule,

$$\frac{dy}{dx}\equiv\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sin t}{1-\cos t}.$$

Substituting this into the DE, the LHS is

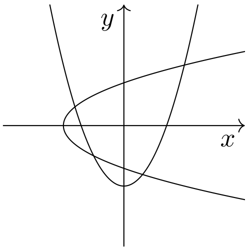
$$\begin{aligned}&\left(\frac{\sin t}{1-\cos t}\right)^2\\ \equiv\ &\frac{\sin^2 t}{(1-\cos t)^2}.\end{aligned}$$

Using the first Pythagorean identity, this is

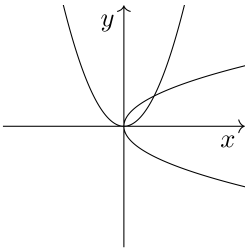
$$\begin{aligned}&\frac{(1-\cos t)(1+\cos t)}{(1-\cos t)^2}\\ \equiv\ &\frac{1+\cos t}{1-\cos t}\\ =\ &\frac{2-y}{y}\\ \equiv\ &\frac{2}{y}-1,\ \text{as required.}\end{aligned}$$

2900. This is not true: there are two such values of a. We can see this by considering three regimes.

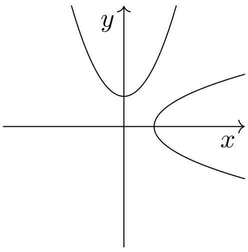
- For large and negative a, the curves have four intersections:



- At $a = 0$, the curves have two intersections:



- For large and positive a, the curves have no intersections:



At the boundary values between these regimes, the curves must be tangent.

— Nota Bene —

This can be seen clearly on a graphing calculator, using a slider to vary the constant a.

— End of 29th Hundred —